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An Innovative Methods of Analysis and Applications of Boolean Algebra in Boolean Matrices

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Abstract— In this paper, we consider a new approach to Boolean Algebra in Boolean Matrices which entails the new method. Basically Boolean Algebra leads us to solve quadratic equation and cubic equation. After converting the equation into Boolean matrices it gives us the solution alternatively by avoiding tedious calculation and lengthy process. This paper gives a solution with this new approach.

Index Terms— Boolean Algebra. Boolean Matrices. Quadratic. Cubic Equations. Binary representation.

1 Introduction

Simple Boolean Quantity: We define a simple Boolean quantity as that which is capable of assuming two values. These values will be denoted by 1 and 0

Examples of Boolean quantities: (a) In logic a proposition can be true or false. The value of a proposition i.e. its truth or falseness is a Boolean quantity. We shall assign arbitrarily the values 1 to true and 0 to false. (b) In a computer, devices with two states are often encountered. e.g. a terminal can have a potential of 0 or 8 V only. The state of this terminal is a Boolean quantity.

Order Relation: We can define a total order relation on the set of values of a simple Boolean variable simply by putting 0 < 1.
We also shall employ the symbols >, $\geqq, \leqq$ with their usual significance.

Duality: We shall give this name to the application of the mapping onto itself of the set of values of a simply Boolean variable, defined by $0 \Rightarrow 1, 1 \Rightarrow 0$
If x is a simple Boolean quantity, we shall denote its dual by x*. It can be verified at once that (x*)* = x. Duality is a symmetric relation.
If $x_1 > x_2$ then $x_1^* < x_2^*$ i.e. duality reverses the order relation in an inequality.

General Boolean quantity: Let E be a finite or infinite set that can be organized in any manner whatever:
Finite case: Column, Matrix.
Infinite case: Sequence of points M_n, Segment.
With each element of this set we associate a simple Boolean quantity. We thus obtain a general Boolean quantity. We denote it as X.

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The simple Boolean quantities which compare them will usually be denoted by the corresponding small letter, with a subscript recalling that it is a corresponding element of the support set, e.g.,

$x_1, x_2, x_3, \ldots\ldots x_n.$

In practice we shall limit ourselves to the case when the support set is finite and the components are written in the column form:

$$X = \left\| \begin{matrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ x_n \end{matrix} \right\|$$

We shall denote by 0 the general Boolean quantity all of whose components are zero, and by I the general Boolean quantity, all of whose components are 1. If desired, the simple Boolean quantities may be regarded as general Boolean quantities with one component.

Spatial representation of a general Boolean quantity: Let X be a general Boolean quantity with n components. Each of the components can take two values so that X has 2^n distinct possibilities. We consider in n-dimensional space the points whose coordinate values are 0 or 1. To each possible X there corresponds one and only one of these points. The points form the vertices of a hypercube.
A segment for n = 1, a square for n = 2, a cube for n = 3. Fig. 1.1 depicts the case n = 3 in perspective.
We have indicated the value of X at certain of the vertices. Representation of the hypercube presents difficulties for $n \Rightarrow 3$, but it continues to offer a convenient geometrical language.
The representations for n = 4 may be seen in Figs. 1.2, 1.3, 1.4. In Fig. 1.2 the last component is 1 for the outer cube, 0 for the inner cube. The other components are located on each of the cubes as in three-dimensional space.

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The eight cubes forming the hypercube may easily be found in Fig. 1.3.
The representation of Fig. 1.4 may even be extended to five variables if desired, Fig. 1.5.

Binary representation of general Boolean quantities: Another convenient method of representing the 2^n values of X consists in writing the components in a row, the 0's and 1's of which they are built up being regarded as binary digits. An integer lying between 0 and 2^n-1 inclusive is thus associated with each value of X. This integer will be termed the affix of the vertex.

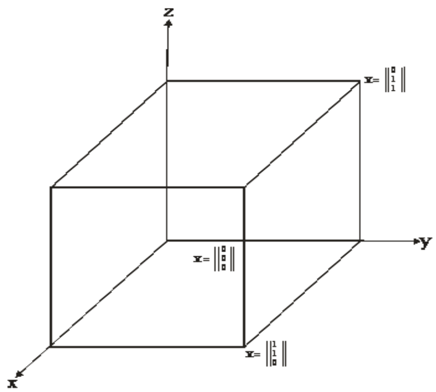

Fig. 1.1

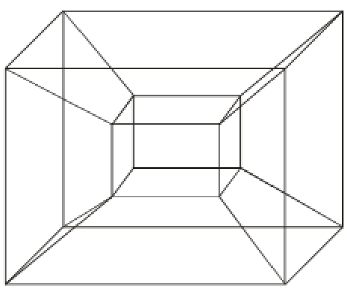

Fig. 1.2

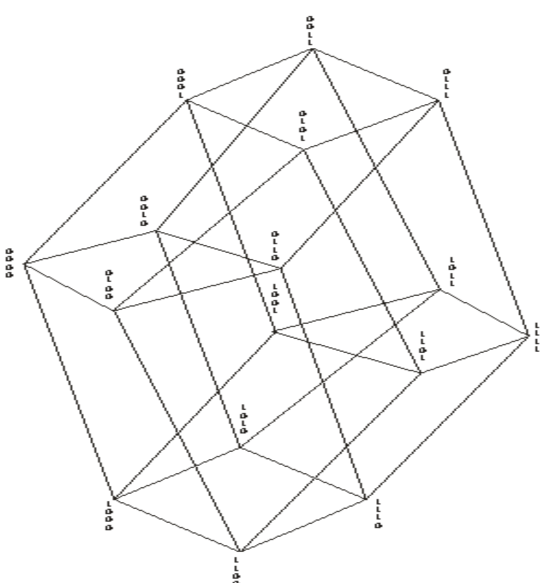

Fig. 1.3

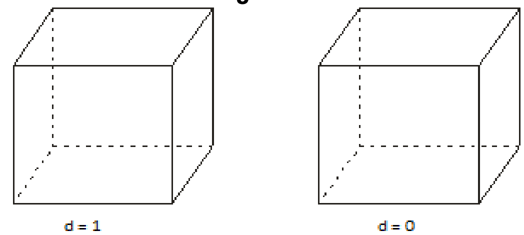

Fig. 1.4

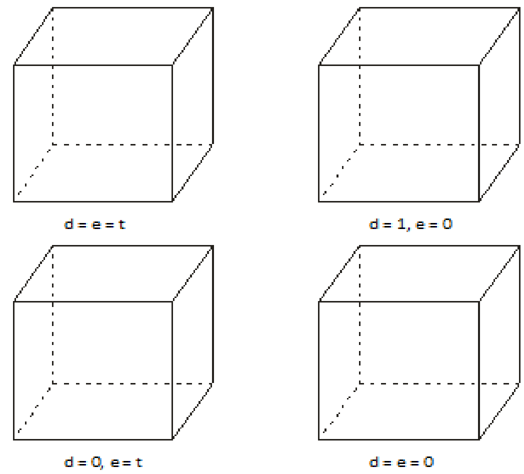

Fig. 1.5

In manuscript, it is convenient to write the affix as a decimal or octal number.

The Fig. 1.6 shows the correspondence, for n = 3 between the vertices of the hypercube and the affixes in decimal form.

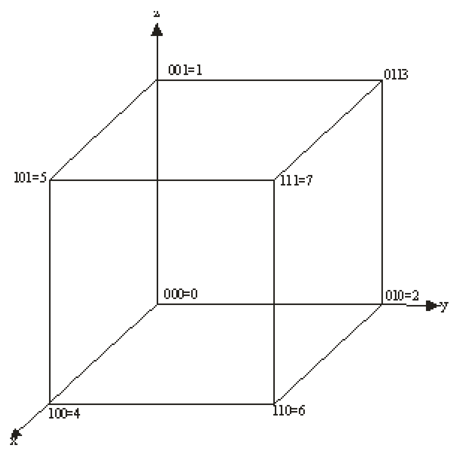

Fig. 1.6

BOOLEAN MATRICES

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Definition (Boolean Matrix): By a Boolean Matrix of size mXn matrix over B_o. Let B_{mn} denote the set of all mXn such matrices. If m = n we write B_n. Elements of B_{mn} are often called relation matrices, Boolean relation matrices, binary relation matrices, binary Boolean matrices, (0, 1)– Boolean matrices and (0, 1) matrices.

Examples:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Definition (Row, Column of Matrix): Let A = (a_{ij}) B_{mn}. Then the element a_{ij} is called the (i, j) entry of A. The (i, j) entry of A is sometimes designated by A_{ij}. The i_{th} row of A is the sequence $a_{i1}, a_{i2},\ldots a_{in}$ and j^{th} column of A is the sequence $a_{1j}, a_{2j},\ldots a_{mj}$. Let A_{i*} and A_{*i} denote the i^{th} row and i^{th} column of A respectively.

In the above Example,

$$A_{1*} = [1\ 1\ 0] \text{ and } A_{*3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Definition: (Zero Matrix, Identity Matrix, Universal Matrix) The n x m zero matrix 0 is the matrix all of whose entries are zero. The n X n identity matrix I (δ_{ij}) such that $\delta_{ij} = 1$ if i = j and $\delta_{ij} = 0$ if $i \neq j$. The n x m universal matrix J is the matrix all of whose entries are 1.

Zero Matrix:

Identity Matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Universal Matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Note: Every identity matrix is Boolean Matrix. But it is not suitable for zero and Universal Matrices.

QUADRATIC EQUATIONS

Theorem: The Boolean matrix J $\ominus$ I has a square root if and only if its dimension is atleast 7 or is 1.

Proof: Let M be a square root of J $\ominus$ I of dimension less than 7. Then all diagonal entries of M are zero, and at least one of m_{ij}, m_{ji} is zero for each i, j. thus for each i the three sets {i}, {j:m_i = 1}, {j : m_{ji} = 1} are disjoint. So one of the latter two sets contains at most two elements. By possibly transposing M, assume that the third set has at most two elements.

If {j : m_{ji} = 1} = {a, b} then $m_{ai} = 1$ and $m_{ji} = 0$ for $j \neq a, b$. Moreover $i \neq a$ and $i \neq b$ since all diagonal entries are zero. Since M^2 = J $\ominus$ I we have $\Sigma m_{ax}.m_{xi} = 1$. But the only non-zero term of this sum is $m_{ab} . m_{bi}$. Thus $m_{ab} = 1$. Also $\Sigma m_{ax}.m_{xi} = 1$. Its only non-zero term is $m_{ba} . m_{ab}$. Thus $m_{ab} = m_{ba} = 1$. But this implies $m_{aa}^{(2)} = 1$ which is false. Likewise the case {j : m_{ji} = 1} = {a} and is equal to $\ominus$ are impossible, unless the dimension is 1.

For large values of n, there are circulants which are square roots of J $\ominus$ I. if S is the set of I's in the last row of a circulant M, M^2 will be J $\ominus$ I if and only if {a+b : a, b $\in$ S} includes numbers in every congruence class modulo n except 0.

If n is odd and at least 9, S = $\left\{1, 2, \ldots\ldots\frac{n}{2}-2, \frac{n}{2}, \frac{n}{2}+2\right\}$ will do. If n = 7, S = 1, 2, 4 will do.

If n is even and atleast 12, let S = $\left\{1, 2, \ldots\ldots\frac{n}{2}-3, \frac{n}{2}-1, \frac{n}{2}+2\right\}$. For n = 8, 10 we use a different construction.

$$\begin{bmatrix} 0 & 1 & 0 & o \\ 0 & & & 1 \\ 1 & & & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where the top and bottom diagonal blocks are 1 x 1, the edge blocks are entirely 1 or entirely 0 as indicated, and the inner four blocks are circulants. For both 8, 10 we can solve to find appropriate circulants.

For n = 7, M is unique up to conjugation by a permutation. However this is not so far large n. This completes the proof. For more details about the square root of J $\ominus$ I see Raghavan [5].

Example: When n is even and atleast 12,

Let S = {1, 2, 3, 5, 8}.

$$Q_{12} = \begin{bmatrix} 0&1&1&1&0&1&0&0&1&0&0&0 \\ 0&0&1&1&1&0&1&0&0&1&0&0 \\ 0&0&0&1&1&1&0&1&0&0&1&0 \\ 0&0&0&0&1&1&1&0&1&0&0&1 \\ 1&0&0&0&0&1&1&1&0&1&0&0 \\ 0&1&0&0&0&0&1&1&1&0&1&0 \\ 0&0&1&0&0&0&0&1&1&1&0&1 \\ 1&0&0&1&0&0&0&0&1&1&1&0 \\ 0&1&0&0&1&0&0&0&0&1&1&1 \\ 1&0&1&0&0&1&0&0&0&0&1&1 \\ 1&1&0&1&0&0&1&0&0&0&0&1 \\ 1&1&1&0&1&0&0&1&0&0&0&0 \end{bmatrix}$$

$Q_{12} \times Q_{12} = (J - I)_{12}$

$$\begin{bmatrix} 0&1&1&1&1&1&1&1&1&1&1&1 \\ 1&0&1&1&1&1&1&1&1&1&1&1 \\ 1&1&0&1&1&1&1&1&1&1&1&1 \\ 1&1&1&0&1&1&1&1&1&1&1&1 \\ 1&1&1&1&0&1&1&1&1&1&1&1 \\ 1&1&1&1&1&0&1&1&1&1&1&1 \\ 1&1&1&1&1&1&0&1&1&1&1&1 \\ 1&1&1&1&1&1&1&0&1&1&1&1 \\ 1&1&1&1&1&1&1&1&0&1&1&1 \\ 1&1&1&1&1&1&1&1&1&0&1&1 \\ 1&1&1&1&1&1&1&1&1&1&0&1 \\ 1&1&1&1&1&1&1&1&1&1&1&0 \end{bmatrix}$$

and hence the theorem.

CUBIC EQUATIONS

Theorem: The Boolean matrix $J \ominus I$ has a cube root which is symmetric circulant whenever its dimension is at least 30.

Proof: Let n denote the dimension.
If $n \equiv k$ (mod 5,0, k = 0, 1, 3 take the circulant whose last row has ones in locations congruent to ± 1, $\pm(5a + 3)$ for all non-negative integers a such that $5a + 3 < \frac{n}{2}$.

If n = 10x + 17, take the circulant whose row has ones in locations 1, 3, 8, ….., 3+5(x-1), 3+5x, 5+5x, 105x, n-(10+5x),…. (n-1), where x is the largest integer such that $3+5x < \frac{n}{3}$. This proves the theorem.

Example: K = 3 n = 33 (mod 5)

$Q_{33} =$

$$\begin{array}{*{33}{c}}
0&1&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&1 \\
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\end{array}$$

$Q_{33} \times Q_{33} =$

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0&1&1&1&1&0&1&0&1&1&0&1&0&1&1&1&1&0&1&1&1&1&0&1&0&1&0&1&0&1&1&1&1 \\
1&0&1&1&1&1&0&1&0&1&1&0&1&0&1&1&1&1&0&1&1&1&1&0&1&0&1&0&1&0&1&1&1 \\
1&1&0&1&1&1&1&0&1&0&1&1&0&1&0&1&1&1&1&0&1&1&1&1&0&1&0&1&0&1&0&1&1 \\
1&1&1&0&1&1&1&1&0&1&0&1&1&0&1&0&1&1&1&1&0&1&1&1&1&0&1&0&1&0&1&0&1 \\
1&1&1&1&0&1&1&1&1&0&1&0&1&1&0&1&0&1&1&1&1&0&1&1&1&1&0&1&0&1&0&1&0 \\
0&1&1&1&1&0&1&1&1&1&0&1&0&1&1&0&1&0&1&1&1&1&0&1&1&1&1&0&1&0&1&0&1 \\
1&0&1&1&1&1&0&1&1&1&1&0&1&0&1&1&0&1&0&1&1&1&1&0&1&1&1&1&0&1&0&1&0 \\
0&1&0&1&1&1&1&0&1&1&1&1&0&1&0&1&1&0&1&0&1&1&1&1&0&1&1&1&1&0&1&0&1
\end{array}$$

$B_{33} = Q_{33} \times Q_{33} \times Q_{33} = (J - I)_{33}$
Hence the theorem.

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